A Simple Expression for Mill's Ratio of the Student's t-Distribution

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Abstract

I show a simple expression of the Mill's ratio of the Student's t-Distribution. I use it to prove Conjecture 1 in Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Mach. Learn.*, 47(2-3): 235–256, May 2002.

We first need the following technical lemma.

Lemma 1. *For all $x > 0$ we have*

$$x\frac{e^{\frac{x^2}{2}}}{\sqrt{e^{x^2}-1}} \leq x+1$$

Proof. Denote by $y = e^{x^2}$, with $y > 1$. We have

$$\frac{\sqrt{y}}{\sqrt{y-1}} - 1 = \frac{\sqrt{y}-\sqrt{y-1}}{\sqrt{y-1}} = \frac{1}{\sqrt{y-1}(\sqrt{y}-\sqrt{y-1})} \leq \frac{1}{\sqrt{y-1}} \,. \tag{1}$$

Hence we have

$$x\left(\frac{e^{\frac{x^2}{2}}}{\sqrt{e^{x^2}-1}} - 1\right) \leq x\frac{1}{\sqrt{e^{x^2}-1}} \leq 1, \tag{2}$$

where in the last inequality we used $\exp(z) - 1 \geq z$. ☐

The following theorem provides simple upper bounds to the Mill's ratio of a t-Student.

Theorem 1. *Let $f_\nu(x)$ the pdf of a Student's t-distribution with ν degrees of freedom. Then, for any $\nu \geq 0$, we have*

$$\frac{\int_a^{+\infty} f_\nu(x)dx}{f_\nu(a)} \leq \sqrt{1+\frac{a^2}{\nu}}\left(\frac{1}{2}+\frac{1}{\sqrt{\nu}}\right), \text{ if } a \geq 0$$

$$\frac{\int_a^{+\infty} f_\nu(x)dx}{f_\nu(a)} \leq \sqrt{1+\frac{a^2}{\nu}}\left(1+\frac{1}{\sqrt{\nu}}\right), \text{ if } a<0$$

Proof. The first stated inequality holds for $a=0$, for the symmetry of the t-Student distribution, hence we can safely assume $a \neq 0$. We have that

$$P[X \geq a] = C_\nu \int_a^{+\infty} \left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx,$$

where $C_\nu = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}$. With the change of variable $z=\sqrt{\nu \log(1+\frac{x^2}{\nu})}$, we have

$$\begin{aligned}
P[X \geq a] &= C_\nu \int_{\sqrt{\nu \log(1+\frac{a^2}{\nu})}}^{+\infty} e^{-\frac{z^2}{2}} \frac{z e^{\frac{z^2}{2\nu}}}{\sqrt{\nu(e^{\frac{z^2}{\nu}}-1)}} dz \\
&\leq C_\nu \int_{\sqrt{\nu \log(1+\frac{a^2}{\nu})}}^{+\infty} e^{-\frac{z^2}{2}} \left(\frac{z}{\sqrt{\nu}}+1\right) dz \\
&= C_\nu \left(\frac{1}{\sqrt{\nu}}\left(1+\frac{a^2}{\nu}\right)^{-\frac{\nu}{2}} + \int_{\sqrt{\nu \log(1+\frac{a^2}{\nu})}}^{+\infty} e^{-\frac{z^2}{2}} dz\right),
\end{aligned}$$

where in the inequality we used Lemma 1. We now use the facts that $\int_a^{+\infty} e^{-\frac{x^2}{2}} dx \leq \frac{1}{2} e^{-\frac{a^2}{2}}$ for $a \geq 0$ and $\int_a^{+\infty} e^{-\frac{x^2}{2}} dx \leq e^{-\frac{a^2}{2}}$ for $a \leq 0$ to have the stated bounds. □

The following Corollary is a slightly better version of Conjecture 1 in Auer et al. [2002]. I could not find a proof of it in any paper so I decided to give a simple proof for it.

Corollary 1. *Let X be a Student's t random variable with ν degrees of freedom. Then, for $0 \leq a \leq \sqrt{2(\nu+1.22)}$ and $\nu \geq 0$, we have*

$$P[X \geq a] \leq e^{\frac{-a^2}{4}}.$$

Proof. First observe that $C_\nu\left(\frac{1}{2}+\frac{1}{\sqrt{\nu}}\right) \leq K = 0.543$, as it can be verified numerically. Using the first result of Theorem 1 we have

$$P[X \geq a] \leq K e^{-\frac{\nu}{2}\log\left(1+\frac{a^2}{\nu}\right)} = e^{\log(K)-\frac{\nu}{2}\log\left(1+\frac{a^2}{\nu}\right)} \leq e^{\log(K)-\frac{a^2\nu}{a^2+2\nu}},$$

where in the last inequality we used the fact that $\log(x+1) \geq \frac{2x}{x+2}$. Hence the statement of the theorem is equivalent to find the upper bound on a^2 such that

$$\log(K) - \frac{a^2\nu}{a^2+2\nu} \leq -\frac{a^2}{4},$$

that in turn is equivalent to

$$a^4 + 2a^2(2\log(K)-\nu) + 8\nu\log(K) \leq 0$$

Solving the quadratic equation we have

$$a^2 \leq \nu - 2\log(K) + \sqrt{\nu^2 + 4\log^2(K) - 12\nu\log(K)} \tag{3}$$

Hence, the condition $a^2 \leq 2\nu - 4\log(K)$ satisifies the inequality above. Using the value of K we have the stated bound. $\square$

References

Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multi-armed bandit problem. *Mach. Learn.*, 47(2-3):235–256, May 2002.